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MATHEMATICAL MODEL OF ULTRASOUND TRANSMISSION IN GRADIENT MATERIALS SYNTHESIZED IN ARC SURFACING UNDER CONTACT LOADING

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Abstract. The paper deals with a two-dimensional problem of time-dependent elasticity theory, examining waves generated on the top layer interface by the source of normal stresses applied to the top layer interface. Here the lower layer interface is free of stresses. The use is made of integral transformations to relate displacement transforms with stresses on the top layer interface. Exponent power series expansion is obtained for the transform of vertical displacements. Analytical approach to each member of the series ensures a precise solution of the problem. Displacement field vs. time dependence on the layer interface has been calculated. The findings are referred to in the process of ultrasound controlling the layer by finite size sensors.

Keywords: stress; deformation; strengthened layer; surfacing; ultrasound waves.

1. Introduction

Arc surfacing is a procedure applied for wear protection of large-sized products [1, 2]. The point of it is coating of the product surface with a wear-resistant overlay via welding electrode melting. As a result, product structure and properties become depth gradient. On the interface “surfaced layer/substrate”, there are mechanical stresses, arising due to the difference and resulting in overlay splitting off. On the other hand, calculated stress-strain state of the overlay in static contact loading [3] indicates compressive stresses on these boundaries, making it possible to conclude that harder overlays further not varying propagation of stresses on the interface of contacting layers. However, the overlays are subject to time-dependent contact loading in the process of operation, as a consequence, temporal variation of stress-strain state is possible. As known, elastic waves appear in a material under such loading and might cause both failure and healing of diverse defects [4]. Therefore, studies on specifics of wave propagation in multilayered materials have been gaining particular importance. The papers are focused on wave propagation in these materials [4 – 8]. The authors [4] investigate Rayleigh wave propagation on the interface “solid/fluid”. As found out, it is relevant for Rayleigh wave velocity whether there is fluid, and it also depends on the wave trajectory curvature, as well on the curvature in the direction perpendicular to the trajectory. The scientists [5, 6] offered a general theory of seismic wave field propagation in multilayered materials and applied it to wave interference in these materials. Pham Chi Vinh et al. [7] explored transmission of elastic waves in the system “functionally gradient material layer/elastic layer”. As a result, reflection and transmission coefficients were obtained explicitly; they were in line with findings of numerical methods. The paper [8] studies Love wave propagation using Wentzel–Kramers–Brillouin method. The dispersion equation for surface Love waves is in ordinary mathematical form. As revealed, coefficients of gradient members influence significantly dispersion curves and phase velocities of Love waves.

Volumetric wave (Lamb wave) transmission in heterogeneous materials is explored in papers [9, 10]. A.V. Avershieva et al [9] used numerical and analytical methods to examine Lamb wave propagation in elastic isotropic and orthotropic layers. They offered a procedure of surface Lamb wave generation with the help of finite elements method and carried out a comparative analysis of a fundamental symmetric mode of Lamb waves around the second critical velocity, determined numerically and analytically. It has been revealed that the growing Poisson's ratio is associated with the monotonous decline of the relative second critical velocity. The layer thickness is of significant importance for dispersion curves corresponding with the fundamental symmetric mode. The review of research procedures to explore Lamb wave transmission in anisotropic materials is provided in [10]. This paper is concerned with six-dimensional Couchy formalism. It helped to form dispersion curves of these waves and carry out a comparative study on the results of numerical methods. It is in a reasonably good agreement with numerical calculations.

The main drawback of investigations into elastic wave transmission, as stated in papers [4 – 10], is insufficient information about peculiarities of wave transmission in a particular layer. A great number of works pay attention to time-dependent load; some of them are of significant importance [11 – 12]. The authors [11] applied the method of integral transformations to solve two-dimensional problem analytically and identify influence of time-dependent load on the elastic layer surface in conditions of mixed boundary value problem on each interface. As a result, normal stress propagation has been obtained in different instants of time and at different distances from the source. Their analysis has pointed out clear stress steps on the symmetry axis; and a forward and the first reflected widening wave have a form of a single jog. Under further reflection the behavior of stress propagation through the layer depth differs from the stepped one due to the distortion wave influence and gradual spreading of wave energy in the cross-sectional direction. When moving away from the source there is smooth stress propagation. The authors [12] investigated the effect on time-dependent load moving on the surface of a homogenous isotropic elastic semis-pace. For uniform motion quadrature solutions were obtained and normal motion propagation of the semi-space surface was analyzed in diverse ranges of load motion speed.

Therefore, this work aims at obtaining solutions of dynamic elasticity theory equations, which are appropriate for ultrasound control of the layer by finite size sensors.

2. Problem statement

We consider the motion problem of waves emitted by the source with a finite action radius r (Fig. 1). Wave receiver is in the point M at the finite length L. We use non-dimensional variables to write the equations:

$$(\bar{x},\bar{z}) = (x,z)/h, \bar{t} = c_1 t/h, (\bar{u},\bar{w}) = (u,w)/h, (\bar{\sigma}_z,\bar{\sigma}_x,\bar{\tau}) = (\sigma_z,\sigma_x,\tau)/(c_1^2\rho) \quad (1)$$

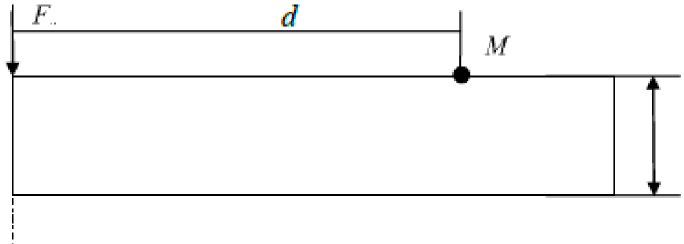

Fig. 1. Statement of the wave propagation problem in the finite thickness layer.

For variables (1) generalized Hooke's law and motion equation are written in the form:

$$\sigma_x = \frac{\partial u}{\partial x} + (1-2c^2)\frac{\partial w}{\partial z}, \sigma_z = (1-2c^2)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}, \tau = c^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right);$$

$$\frac{\partial^2 u}{\partial x^2} + (1-c^2)\frac{\partial^2 w}{\partial x \partial z} + c^2\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad (2)$$

$$c^2\frac{\partial^2 w}{\partial x^2} + (1-c^2)\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial t^2} = 0,$$

where $c^2 = \frac{c_2^2}{c_1^2}, c_1^2 = \frac{2\mu+\lambda}{\rho}, c_2^2 = \frac{\mu}{\rho}$, λ and μ – Lame parameters, ρ – material density. The conditions on the layer interface are in the form:

$$z = 1: \sigma_z = -\frac{1}{\pi}\frac{\delta}{\delta^2 + x^2}\big(\Theta(t) - \Theta(t - T_0)\big), \sigma_{xz} = 0;$$

$$z = -1: \sigma_z = \sigma_{xz} = 0, \quad (3)$$

where Θ – Heaviside function, T_0 – signal response time.

3. Results of calculations

We use the method of finite elements to solve the boundary problem (2), (3). Problem parameters are given in Table 1.

Table 1. Problem parameters.

Symbol	Description	Value
d	Source to receiver distance	20 mm
H	Plate thickness	50,15, 10, 3 mm
F_y	Source force amplitude	100 H
t_0	Pulse duration	$2 \cdot 10^{-6}$ s
E	Young modulus	70 GPa
V	Poisson's ratio	0.33
Δx	Width of load pulse	1 mm

A fit for vertical loading (3) was made using Fourier method to ease the calculations. Longitudinal (a) and time (b) propagation of load from the source is given in Fig. 2.

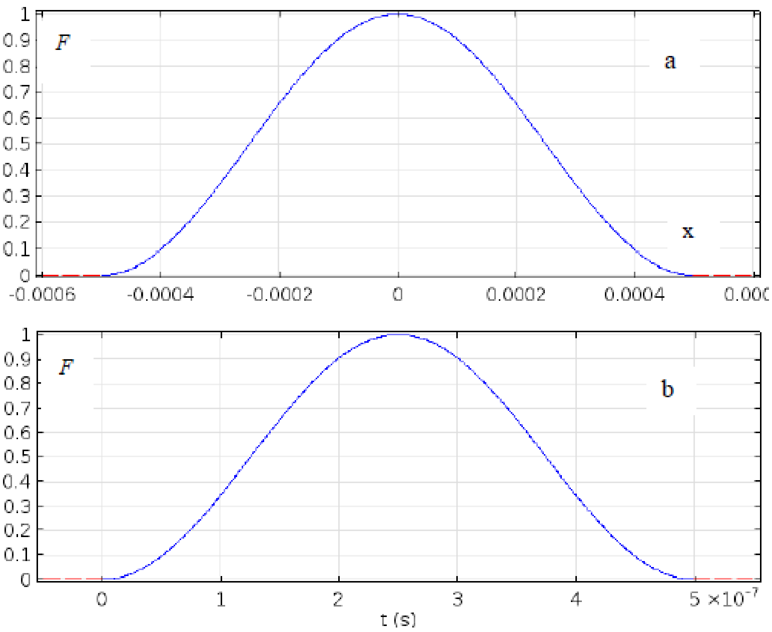

Fig. 2. Load propagation from the source.

The calculation data of cross-sectional displacement velocity fields in the point M at a plate thickness of 50 mm is given in Fig. 3. Their analysis demonstrates the similarity of these plates to Raleigh wave propagation. The situation gets more complicated when layer thinning (Fig. 4).

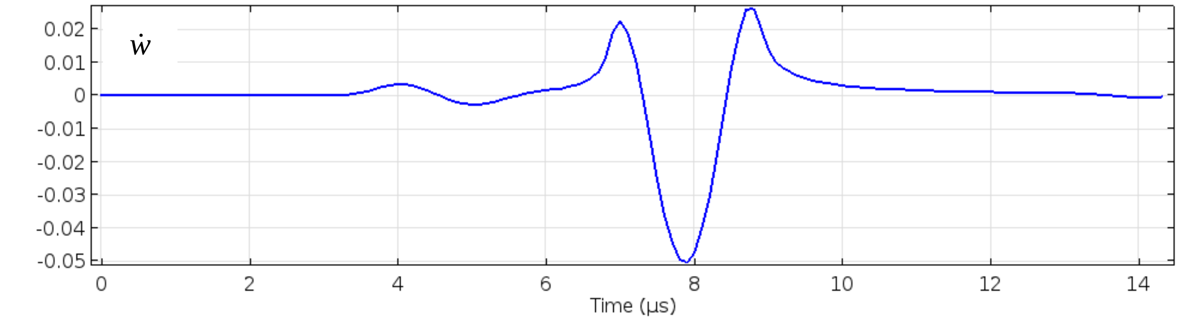

Fig 3. Calculation data of velocity fields for a 50 mm thick plate.

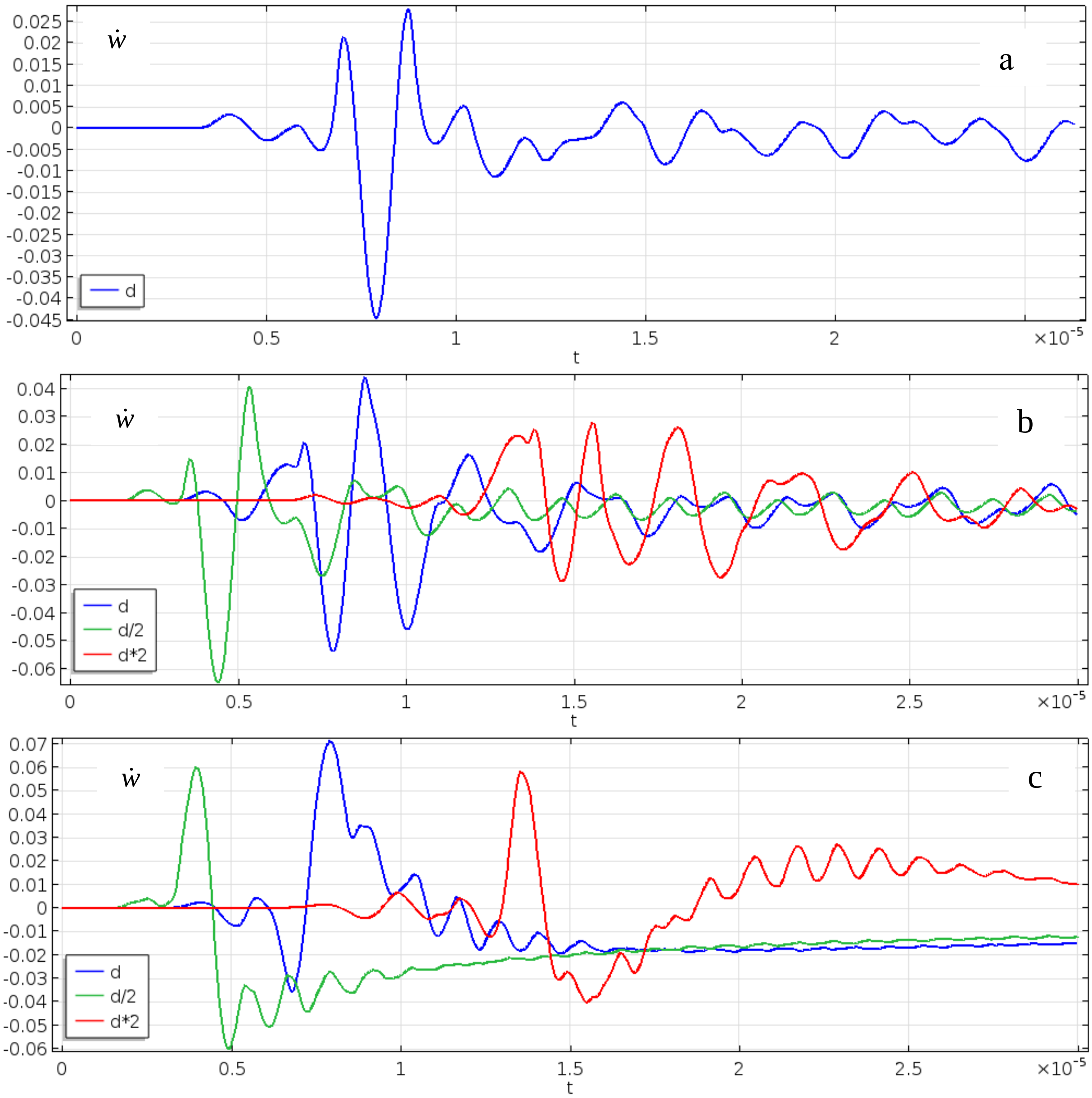

Fig. 4. Calculation data of velocity fields for 15 mm (a), 10 mm (b), 3 mm (c) plates.

The observed wave pattern is a result of travelling pulse overlapping and multi-reflected waves from the lower boundary. It is noteworthy, that after pulse termination in a 3 mm thick plate (Fig. 4c) the displacement velocity gets stabilized in the range of negative values around the value of – 0.015, whereas at the distance of 2d from the source there is strengthening observed in the range of positive values in the interval 1.5 < t < 2 μs, and declining to 0.01 at t > 2.5 μs. This fact indicates volumetric Lamb waves in the material alongside with the surface waves.

4. Results of analytical estimations

We transform (2) and (3) using integral Laplace transform for time and Fourier method for the coordinate. Then (3) is written in the form:

$$\Sigma_1 = -\frac{exp(-\delta|q|)(1-exp(-pT))}{p}, T_1 = T_{-1} = \Sigma_{-1} = 0, \tag{4}$$

and the system of equations (2)

$$W'''' - (n_1^2 + n_2^2)W'' + n_1^2 n_2^2 W = 0, U = -\frac{i}{(1-b^2)qn_1^2}\left(W''' - \left(n_2^2 + n_1^2(1-b^2)\right)W'\right),$$
$$n_1^2 = q^2 + p^2, n_2^2 = q^2 + p^2 b^2, b^2 = 1/c^2, \tag{5}$$
$$\Sigma(z) = W'(z) - i(2c^2 - 1)qU(z),$$
$$T(z) = c^2(U'(z) + iqW(z)).$$

We express stresses on the interfaces with the help of displacement value W on the interfaces and its derivatives

$$\Sigma = -\frac{1}{b^2(b^2-1)n_1^2}\left((b^2-1)W''' - (2(b^2-1)n_1^2 + (b^2-1)n_2^2)W'\right),$$
$$T = \frac{i}{b^2(b^2-1)qn_1^2}(b^2W'' - (2n_2^2 + n_1^2 b^2)W). \tag{6}$$

For displacement transform W we obtain a boundary problem:

$$W'''' - (n_1^2 + n_2^2)W'' + n_1^2 n_2^2 W = 0,$$
$$(2-b^2)W'''(1) - (2(1-b^2)n_1^2 + (2-b^2)n_2^2)W'(1) = -b^2(1-b^2)n_1^2\Sigma_1,$$
$$(2-b^2)W'''(-1) - (2(1-b^2)n_1^2 + (2-b^2)n_2^2)W'(-1) = 0,$$
$$b^2W''(1) - (2n_2^2 - n_1^2 b^2)W(1) = 0,$$
$$b^2W''(-1) - (2n_2^2 - n_1^2 b^2)W(-1) = 0. \tag{7}$$

The solution of the boundary problem (11) is in the form:

$$W(z) = S_1\, sinh(n_1 z) + S_2\, sinh(n_2 z) + C_1\, cosh(n_1 z) + C_2\, cosh(n_2 z). \tag{8}$$

Substituting (8) into (7) results in two systems of algebraic equations:

$$\begin{cases} s_{11}S_1 + s_{12}S_2 = f, \\ s_{21}S_1 + s_{22}S_2 = 0; \end{cases} \begin{cases} c_{11}C_1 + c_{12}C_2 = f, \\ c_{21}C_1 + c_{22}C_2 = 0; \end{cases}$$
$$s_{11} = (n_2^2 + q^2)\, cosh(n_1), s_{12} = 2n_1 n_2\, cosh(n_2),$$
$$S_{21} = 2q^2\, sinh(n_1), s_{22} = (n_2^2 + q^2)\, sinh(n_2), \tag{9}$$
$$c_{11} = (n_2^2 + q^2)\, sinh(n_1), c_{12} = 2n_1 n_2\, sinh(n_2),$$
$$c_{21} = 2q^2\, cosh(n_1), \;\; c_{22} = (n_2^2 + q^2)\, cosh(n_2), f = -b^2 n_1\, \Sigma_1/2.$$

Solution (9):

$$S_1 = \frac{b^2 n_1(n_2^2 + q^2)\sinh(n_2)}{2\Delta_s}\Sigma_1, S_2 = -\frac{b^2 n_1 q^2 \sinh(n_1)}{\Delta_s}\Sigma_1,$$
$$C_1 = \frac{b^2 n_1(n_2^2+q^2)\cosh(n_2)}{2\Delta_c}\Sigma_1, C_2 = -\frac{b^2 n_1 q^2\cosh(n_1)}{\Delta_c}\Sigma_1, \tag{10}$$
$$\Delta_s = (n_2^2 + q^2)^2\cosh(n_1)\sinh(n_2) - 4n_1 n_2 q^2\cosh(n_2)\sinh(n_1),$$
$$\Delta_c = (n_2^2 + q^2)^2\cosh(n_2)\sinh(n_1) - 4n_1 n_2 q^2\cosh(n_1)\sinh(n_2).$$

Using (10) and (8) we write the expression for W(1), which relates stress transforms to those of displacement on the top layer interface:

$$W = \frac{b^2 n_1(n_2^2 - q^2)}{2}\left(\frac{\cosh n_1 \cosh n_2}{\Delta_c} + \frac{\sinh n_1 \sinh n_2}{\Delta_s}\right)\Sigma_1. \tag{11}$$

Taking into account (3) the formula (11) is given in the form:

$$W = -\frac{b^4 p n_1 exp(-\delta|q|)(1-exp(-pT))}{2} L,$$
$$L = \left(\frac{cosh\, n_1\, cosh\, n_2}{\Delta_C} + \frac{sinh\, n_1\, sinh\, n_2}{\Delta_S}\right). \tag{12}$$

To compare with the numerical results a time-correlation of cross-sectional displacements on the upper interface is required at distances $x = l: w(l, h, t) = \Psi(t)$. For this purpose, we transform denominators (11) as follows:

$$\Delta_S = \frac{R_1}{4} exp(n_1 + n_2)(1 - q_1), \Delta_C = \frac{R_1}{4} exp(n_1 + n_2)(1 - q_2),$$
$$R_1 = (n_2^2 + q^2)^2 - 4q^2 n_1 n_2, R_2 = (n_2^2 + q^2)^2 + 4q^2 n_1 n_2, \tag{13}$$
$$q_1 = A + \gamma B, q_2 = A - \gamma B,$$
$$A = exp\big(-2(n_1 + n_2)\big), B = exp(-2n_2) - exp(-2n_1), \gamma = R_2/R_1.$$

We obtain for L:

$$L = \frac{2}{R_1}(1 + F_1 + F_2 + \ldots),$$
$$F_1 = \gamma(\gamma - 1)exp(-4n_1) + \gamma(\gamma + 1)exp(-4n_2) - 2(\gamma^2 - 1)exp(-2n_1 - 2n_2), \tag{14}$$
$$F_2 = -\gamma^3\, exp(-8n_1) + \gamma^3\, exp(-8n_2) - 2(4\gamma^2 - 1)\, exp(-4n_1 - 4n_2) +$$
$$2\gamma(\gamma^2 + 2\gamma - 1)\, exp(-6n_1 - 2n_2) - 2\gamma(\gamma^2 - 2\gamma - 1)\, exp(-2n_1 - 6n_2).$$

We substitute obtained expressions into W_1. It is proper to consider velocities instead of displacements. For velocity transform $\dot{W} = pW$ we have:

$$\dot{W} = -\frac{b^4 p^2 n_1 exp(-\delta|q|)(1-exp(-pT))}{R_1}(1 + F_1 + F_2). \tag{15}$$

Finally, we obtain a sum total, where the first summand is the time, when waves propagating along the surface arrive at the observation point, the second one (F_2) is the time, when single-reflected waves from the back surface arrive at the observation point; the third summand (F_3) is the time of double-reflected waves arrival. The general view of a summand in (15) can be given in the form:

$$W^{LF}(q, p) = \frac{1}{p}\Phi(p, q)exp(-\delta|q| - \alpha n_1 - \beta n_2 - \tau p) = \sigma_0^{LF}(p, q)exp(-\delta|q|). \tag{16}$$

Here $n_{1,2}$ are homogenous first-order functions, Φ is of the zero-order, τ can have a zero or T value. Transforms like (16) are found in more simple dynamic problems, when one of the coefficients α or β is equal to zero. When both of them are not equal to zero, these cases are considered more complicated. The required correlation $\dot{w}(t)$ is written as a sum total of original functions, each of them is obtained using a double inverse transform of expressions like (16). For inverse transform we apply the procedure proposed in [13]. We determine function (L – image of the required function), as in [14]:

$$\sigma^L(p, s) = \frac{1}{\pi} Re \int_0^\infty \sigma_0^{LF}(p, q)\, exp(-sq)\, dq; s = \delta + ix. \tag{17}$$

In (17) we replace $q = \xi p$.

$$\sigma^L(p, s) = \frac{1}{\pi} Re \int_0^\infty \Phi\,(\xi)\, exp\big(-p(\alpha m_1 + \beta m_2 + s\xi + \tau)\big)\, d\xi,$$
$$m_1 = \sqrt{1 + \xi^2}, m_2 = \sqrt{b^2 + \xi^2}, \text{ at } Re\xi > 0, Im\xi = 0. \tag{18}$$

We introduce a new variable t according to the formula:

$$t = \alpha m_1 + \beta m_2 + s\xi - \alpha - \beta b. \tag{19}$$

Function $t(\xi)$ increases monotonically, since the derivative is positive

$$t' = (\xi')^{-1} = \alpha m_1' + \beta m_2' + s > 0 (\xi > 0). \tag{20}$$

Therefore, the equation (19) is assumed as a single-valued function $\xi(t, s)$. Using in (18) the variable t we obtain:

$$\sigma^L(p, s) = \frac{1}{\pi} Re \left(exp\big(-p(\tau + \alpha + b\beta)\big) \int_0^\infty \frac{\Phi\big(\xi(t,s)\big)}{\alpha m_1' + \beta m_2' + s} exp(-pt) d\xi\right),$$
$$m_1 = \sqrt{1 + \xi^2}, m_2 = \sqrt{b^2 + \xi^2};\ m_{1,2} > 0\ at\ Re\xi > 0, Im\xi = 0 \tag{21}$$

Using the displacement theorem of original we get Laplace transform of the function in the right part (21):

$$\sigma(t,s) = \frac{1}{\pi} Re\left(\frac{\Phi\big(\xi(t-(\tau+\alpha+b\beta),s)\big)}{\alpha m_1' + \beta m_2' + s}\right), s = \delta + ix, x > 0. \quad (22)$$

The formula (22) provides an expression for the reversal double Fourier – Laplace transform and is a required function of t, x. We consider the first summand in (19) in the form:

$$\dot{W}_0 = -\frac{b^4 p^2 n_1 exp(-\delta|q|)(1-exp(-pT))}{R_1}. \quad (23)$$

To compare (23) with (16) we have $\alpha = \beta = 0$, and for Φ_0

$$\Phi_0(\xi) = -\frac{b^4 m_1(\xi)}{R_1(\xi)};\ R_1(\xi) = (m_2^2 + \xi^2)^2 - 4\xi^2 m_1 m_2;\ \xi = t/s. \quad (24)$$

Consequently,

$$\dot{w}_0(t,s) = \frac{1}{\pi} Re \frac{1}{s}\Big(\Phi\big(\xi(t,s)\big) - \Phi\big(\xi(t-\tau),s\big)\Big). \quad (25)$$

Using the formula (25), time correlation of the transverse velocity ($\dot{w}_0(t,s)$) is plotted (Fig. 5). The dependence above is similar to that given in Fig. 3 for a 50 mm thick plate, making it possible to conclude that travelling waves will be registered in thick plates at the distances of approximately 20 mm.

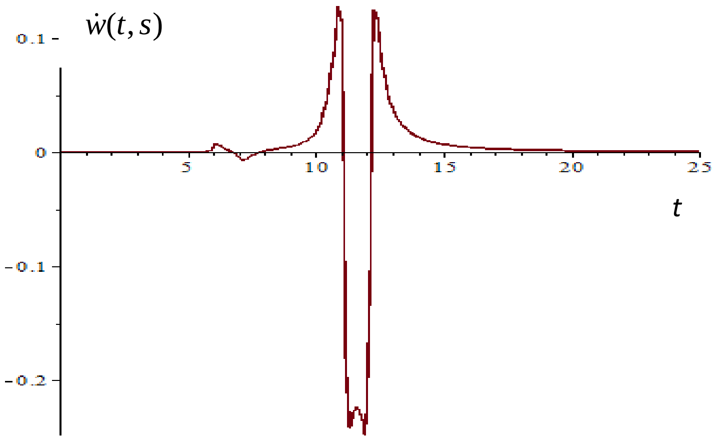

Fig. 5. Transverse velocity ($\dot{w}_0(t,s)$) vs. time dependence at $x = 6;\ b = 1.7;\ \tau = 1;\ \delta = 0.1$.

To take into account reflected waves the second summand in (15) is written in the form $\dot{W}_1^{LF} = \dot{W}_0^{LF} L_1$ it can be presented as three summands provided that $T \to \infty$

$$\dot{W}_{11}^{LF} = -\frac{b^4 p^2 n_1 \gamma(\gamma-1) exp(-\delta|q| - 4n_1)}{R_1},$$
$$\dot{W}_{12}^{LF} = -\frac{b^4 p^2 n_1 \gamma(\gamma+1) exp(-\delta|q| - 4n_2)}{R_1}, \quad (26)$$
$$\dot{W}_{13}^{LF} = \frac{2b^4 p^2 n_1(\gamma^2 - 1) exp(-\delta|q| - 2n_1 - 2n_2)}{R_1}.$$

Therefore, for functions Φ we will have with regard to (14) and replacement $q = \xi p$

$$\Phi_{11}^{LF} = -\frac{8b^4\xi^2 m_1^2 m_2 R_2(\xi)}{R_1^3(\xi)}, \Phi_{12}^{LF} = -\frac{2b^4 m_1 (m_2^2+\xi^2)^2 R_2(\xi)}{R_1^3(\xi)},$$
$$\Phi_{13}^{LF} = \frac{32b^4\xi^2 m_1^2 m_2 (m_2^2+\xi^2)^2}{R_1^3(\xi)}, \quad (27)$$
$$R_1(\xi) = (m_2^2 + \xi^2)^2 - 4\xi^2 m_1 m_2,\ R_2(\xi) = (m_2^2 + \xi^2)^2 + 4\xi^2 m_1 m_2.$$

For each function $\Phi_{1k}(k = 1,2,3)$ its own dependence is to be found out $\xi_k(t,s)$, and for them an equation like (15) is to be obtained. This equation in the general case can be reduced to a biquadratic algebraic equation with complex coefficients, this equation is solved numerically. In the general case of four solutions we select those, which agree with (15), and in the rest – those, which can satisfy with the correlation $Re(\xi(t)) > 0, Im(\xi(t)) < 0$. Using (22) and (26) at $\tau = 0$ we write formulae for single-reflected waves

$$\begin{aligned} \Psi_{11}(t) &= \frac{1}{\pi} Re\left(\frac{\Phi_{11}(\xi)m_1(\xi)}{2 + sm_1(\xi)}\right), \xi = \xi_1(t-4),\\ \Psi_{12}(t) &= \frac{1}{\pi} Re\left(\frac{\Phi_{12}(\xi)m_2(\xi)}{2 + sm_2(\xi)}\right), \xi = \xi_2(t-4b),\\ \Psi_{13}(t) &= \frac{1}{\pi} Re\left(\frac{\Phi_{13}(\xi)m_1(\xi)m_2(\xi)}{2\big(m_1(\xi)+m_2(\xi)\big) + sm_1(\xi)m_2(\xi)}\right), \xi = \xi_3(t-2-2b). \end{aligned} \quad (28)$$

For the resultant signal we obtain $\Psi_1(t) = \Psi_{11}(t) + \Psi_{12}(t) + \Psi_{13}(t)$. As long as $\tau \neq 0$, we have

$$\dot{w}_1(t, x, \delta, \tau) = \Psi_1(t) - \Psi_1(t-\tau). \quad (29)$$

Therefore, analytical solutions have been obtained and can be used to analyze wave propagation in the system “source-receiver” of the finite sizes. For travelling waves they were compared with the numerical results, principal agreement of calculations with analytical solutions was identified for thick plates.

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